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Note

An Application of a Fixed-Point Theorem to Approximation Theory

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In this note, an extension of a theorem of Brosowski is given where linearity of the function and the convexity of the set are relaxed.

Brosowski [1] has proved the following:

THEOREM. *Let T be a contractive linear operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty, compact, and convex, then it contains a T-invariant point.*

A similar theorem will be proved when T is not a linear operator and the set of best C-approximants is not necessarily a convex set.

We need the following definition.

Let X be a linear space. A subset C in X is said to be starshaped if there is a point p in C such that $x \in C$ and $0 \leqslant \lambda \leqslant 1$ implies $\lambda p + (1 - \lambda)x \in C$.

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Proof of the Theorem. Let D be the set of best C-approximants to x. Then $T: D \to D$ (since, if $y \in D$, then $\| Ty - x \| = \| Ty - Tx \| \leqslant \| y - x \|$, then $Ty \in D$).

Take $p \in D$ such that $\lambda p + (1 - \lambda)x \in D$ for all $x \in D$ and $0 \leqslant \lambda \leqslant 1$.

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Let k_n, $0 \leqslant k_n < 1$, be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Then define

$$T_n\colon D \to D$$

by $T_n x = k_n Tx + (1 - k_n)p$ for all $x \in D$.

Since T maps D into D, T_n also maps D into D for each n. Also, we have

$$\begin{aligned} \| T_n x - T_n y \| &= k_n \| Tx - Ty \| \\ &\leqslant k_n \| x - y \| \\ &< \| x - y \| \qquad \text{for all } x, \quad y \in D, \quad x \neq y. \end{aligned}$$

Then, since D is compact, T_n has a unique fixed point, say x_n for each n (Edelstein's Theorem [2]). Thus, $T_n x_n = x_n$ for each n. Since D is compact, x_n has a convergent subsequence x_{n_i} converging to $\bar{x}$ say.

We claim that $T\bar{x} = \bar{x}$. Now, $x_{n_i} = T_{n_i} x_{n_i} = (1 - k_{n_i})p + k_{n_i} T x_{n_i}$.

Taking limit as $i \to \infty$, $k_{n_i} \to 1$, we have $\bar{x} = T\bar{x}$. ($x_{n_i} \to \bar{x}$ then $Tx_{n_i} \to T\bar{x}$ as T is continuous.) Thus $\bar{x}$ is a T invariant.

Each convex set is necessarily starshaped, but a starshaped set need not be convex.

References

1. B. Brosowski, Fix punktsatze in der Approximations theorie, *Mathematica (Cluj)* **11** (1969), 195–220.
2. M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.* **37** (1962), 74–79.